M.Math. IInd year
First semestral exam 2016
Number Theory
Instructor : B.Sury
Attempt ONLY SIX questions; Maximum Marks 60.

Q 1. (10 marks)
Characterize with proof all primes expressible as $a^2 + 2b^2$.

OR

Let $p > 2$ be a prime and suppose a is a primitive root mod p. Prove that either a or $a + p$ is a primitive root mod p^n for all $n > 1$.

Q 2. (10 marks)
Prove that a Fermat number $2^{2^n} + 1$ can never be a prime power p^r with $r \geq 2$.

OR

Prove that $(p-2)! = 1 + p^n$ does not have any solution in primes $p > 5$ and positive integers n.

Q 3. (10 marks)
If n is not a prime, show that the sequence $d(n), d(d(n)), d(d(d(n))), \cdots$ must contain a perfect square.

OR

Prove that $\sum_{r|n} d(r)^3 = (\sum_{r|n} d(r))^2$.

Q 4. (12 marks)
Let a, b be coprime integers and c be a positive integer. Use the Chinese remainder theorem to prove there exists an integer n such that $a + nb$ is coprime to c.

OR

Let $\zeta = e^{2i\pi/n}$ and $Q(n) = \prod_{(d,n)=1}(1 - \zeta^d)$. Using the given hint or otherwise, prove for $n > 1$ that $Q(n) = p$ or 1 according as to whether n is a power of a prime p or not a prime power.
Hint: Deduce $P(n) := \prod_{r=1}^{n-1}(1 - \zeta^r) = n = \prod_{d|n} Q(d)$ and use the Möbius inversion to deduce $Q(n) = \prod_{d|n} d^{\mu(n/d)}$ and use this to complete the solution.

Q 5. (12 marks)
Prove that the quadratic form $7x^2 + 25xy + 23y^2$ takes the same values as the quadratic form $x^2 + xy + 5y^2$ over integers.

OR

If $ax^2 + bxy + cy^2$ is a reduced, positive-definite integral form and $au^2 + buv + cv^2 \leq a + |b| + c$ for some $(u, v) = 1$, prove that $au^2 + buv + cv^2$ must be one of $a, c, a \pm |b| + c$.

Q 6. (10 marks)
Obtain the value of the periodic, simple, continued fraction $[1; \overline{2, 3}]$.

OR

Obtain the simple, continued fraction expansion of $\sqrt{a^2 + 1}$ for any natural number a.

Q 7. (12 marks)
(i) For any positive integer n, prove that $\{1, 2, \cdots, 2n\}$ is a union of n pairs whose sums are primes.
(ii) If $\psi(n) := \sum_{p \ prime, k \geq 1, p^k \leq n} \log(p)$, prove that $\psi(n) \geq n \log(2)$ for sufficiently large n.

OR

(i) Prove that LCM $(1, 2, \cdots, n) \geq 2^n$ for $n \geq 7$.
(ii) Prove that for $n > 1$, $n!$ is not a k-th power for any $k > 1$.